

The edge Folkman number $F_e(3, 3; 4)$ is greater than 19

Aleksandar Bikov* Nedyalko Nenov

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd.
1164 Sofia, Bulgaria

September 13, 2016

Abstract

The set of the graphs which do not contain the complete graph on q vertices K_q and have the property that in every coloring of their edges in two colors there exist a monochromatic triangle is denoted by $\mathcal{H}_e(3, 3; q)$. The edge Folkman numbers $F_e(3, 3; q) = \min\{|V(G)| : G \in \mathcal{H}_e(3, 3; q)\}$ are considered. Folkman proved in 1970 that $F_e(3, 3; q)$ exists if and only if $q \geq 4$. From the Ramsey number $R(3, 3) = 6$ it becomes clear that $F_e(3, 3; q) = 6$ if $q \geq 7$. It is also known that $F_e(3, 3; 6) = 8$ and $F_e(3, 3; 5) = 15$. The upper bounds on the number $F_e(3, 3; 4)$ which follow from the construction of Folkman and from the constructions of some other authors are not good. In 1975 Erdős posed the problem to prove the inequality $F_e(3, 3; 4) < 10^{10}$. This Erdős problem was solved by Spencer in 1978. The last upper bound on $F_e(3, 3; 4)$ was obtained in 2012 by Lange, Radziszowski and Xu, who proved that $F_e(3, 3; 4) \leq 786$. The best lower bound on this number is 19 and was obtained 10 years ago by Radziszowski and Xu. In this paper, we improve this result by proving $F_e(3, 3; 4) \geq 20$. At the end of the paper, we improve the known bounds on the vertex Folkman number $F_v(2, 3, 3; 4)$ by proving $20 \leq F_v(2, 3, 3; 4) \leq 24$.

Keywords: Folkman number, clique number, independence number, chromatic number

1 Introduction

In this paper only finite, non-oriented graphs without loops and multiple edges are considered. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Let G_1 and G_2 be two graphs. Then, we denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) =$

*Corresponding author

†Email addresses: asbikov@fmi.uni-sofia.bg, nenov@fmi.uni-sofia.bg

$E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$, i.e. G is obtained by making every vertex of G_1 adjacent to every vertex of G_2 . The Ramsey number is denoted by $R(p, q)$. More detailed information on the Ramsey numbers, which we use in this paper, can be found in [20]. All undefined terms and notations can be found [28].

The expression $G \xrightarrow{e} (3, 3)$ means that in every coloring of the edges of the graph G in two colors there is a monochromatic triangle. It is well known that $K_6 \xrightarrow{e} (3, 3)$. Therefore, if $\omega(G) \geq 6$, then $G \xrightarrow{e} (3, 3)$. In [4] Erdős and Hajnal posed the following problem: *Is there a graph $G \xrightarrow{e} (3, 3)$ with $\omega(G) < 6$?* A positive answer was given by several authors, and Folkman [5] was the first to construct such a graph with clique number 3.

Later, we will use the following fact:

$$(1.1) \quad G \xrightarrow{e} (3, 3) \Rightarrow \chi(G) \geq 6,$$

which is a special case of a more general result from [10].

Denote:

$$\mathcal{H}_e(3, 3; q) = \left\{ G : G \xrightarrow{e} (3, 3) \text{ and } \omega(G) < q \right\}.$$

$$\mathcal{H}_e(3, 3; q; n) = \{ G : G \in \mathcal{H}_e(3, 3; q) \text{ and } |V(G)| = n \}.$$

The edge Folkman number $F_e(3, 3; q)$ is defined with

$$F_e(3, 3; q) = \min \{ |V(G)| : G \in \mathcal{H}_e(3, 3; q) \}$$

According to Folkman's result [5], $F_e(3, 3; q)$ exists if and only if $q \geq 4$.

Since $K_6 \xrightarrow{e} (3, 3)$ and $K_5 \not\xrightarrow{e} (3, 3)$, $F_e(3, 3; q) = 6, q \geq 7$.

In 1968 Graham [7] obtained the equality $F_e(3, 3; 6) = 8$ by proving that $K_3 + C_5 \xrightarrow{e} (3, 3)$.

Nenov [15] constructed a 15-vertex graph in $\mathcal{H}_e(3, 3; 5)$ in 1981, thus proving $F_e(3, 3; 5) \leq 15$. In 1999 Piwakowski, Radziszowski and Urbański [19] completed the computation of the number $F_e(3, 3; 5)$ by proving with the help of a computer that $F_e(3, 3; 5) \geq 15$. They also obtained all graphs in $\mathcal{H}_e(3, 3; 5; 15)$.

The graph G obtained by the construction of Folkman, for which $G \xrightarrow{e} (3, 3)$ and $\omega(G) = 3$, has a very large number of vertices. Because of this, in 1975 Erdős [3] posed the problem to prove the inequality $F_e(3, 3; 4) < 10^{10}$. In 1986 Frankl and Rödl [6] almost solved this problem by showing that $F_e(3, 3; 4) < 7.02 \times 10^{11}$. In 1988 Spencer [27] proved the inequality $F_e(3, 3; 4) < 3.10^9$, after an erratum by Hovey, by using probabilistic methods. In 2008 Lu [11] constructed a 9697-vertex graph in $\mathcal{H}_e(3, 3; 4)$, thus considerably improving the upper bound on $F_e(3, 3; 4)$. Soon after that, Lu's result was improved by Dudek and Rödl [2], who proved $F_e(3, 3; 4) \leq 941$. The best upper bound known on $F_e(3, 3; 4)$ was obtained in 2012 by Lange, Radziszowski and Xu [9], who constructed a 786-vertex graph in $\mathcal{H}_e(3, 3; 4)$, thus showing that $F_e(3, 3; 4) \leq 786$.

In 1972 Lin [10] proved that $F_e(3, 3; 4) \geq 11$. The lower bound was improved by Nenov [16], who showed in 1981 that $F_e(3, 3; 4) \geq 13$. In 1984 Nenov [17] proved that every 5-chromatic K_4 -free graph has at least 11 vertices, from which it is easy to derive that $F_e(3, 3; 4) \geq 14$. From $F_e(3, 3; 5) = 15$ [19] it follows easily, that $F_e(3, 3; 4) \geq 16$. The best lower bound known on $F_e(3, 3; 4)$ was obtained in 2007 by Radziszowski and Xu [21], who proved with the help of a computer that $F_e(3, 3; 4) \geq 19$. According to Radziszowski and Xu [21], any method to improve the bound $F_e(3, 3; 4) \geq 19$ would likely be of significant interest.

A more detailed view on results related to the numbers $F_e(3, 3; q)$ is given in the book [26], and also in the papers [21], [8], [9] and [22].

In this paper, we improve the lower bound on the number $F_e(3, 3; 4)$ by proving with the help of a computer the following

Main Theorem. $F_e(3, 3; 4) \geq 20$.

At the end of the paper, we consider the vertex Folkman number $F_v(2, 3, 3; 4)$ and prove $20 \leq F_v(2, 3, 3; 4) \leq 24$ (Theorem 4.1). This is an improvement over the known bounds $19 \leq F_v(2, 3, 3; 4) \leq 30$ from [25] and [24].

This paper is organized in 4 sections. In the first section, the necessary notations are introduced, an overview of the related results is given, and the Main Theorem is formulated. In the second section, we present a computer algorithm (Algorithm 2.4), with the help of which in the third section we prove the Main Theorem. In the last fourth section, new bounds on the number $F_v(2, 3, 3; 4)$ are obtained (Theorem 4.1).

2 Algorithm

Let $G \in \mathcal{H}_e(3, 3; 4; n)$, $A \subseteq V(G)$ be an independent set of vertices of G , $|A| = s$ and $H = G - A$. Then, obviously, G is a subgraph of $\overline{K}_s + H$, and therefore $\overline{K}_s + H \in \mathcal{H}_e(3, 3; 5; n)$, from which it is easy to see that $K_1 + H \in \mathcal{H}_e(3, 3; 5; n - s + 1)$. By this reasoning, in [21] it is proved, but not explicitly formulated (see the proofs of Theorem 2 and Theorem 3), the following

Proposition 2.1. *Let $G \in \mathcal{H}_e(3, 3; 4; n)$, $A \subseteq V(G)$ be an independent set of vertices of G and $H = G - A$. Then, $K_1 + H \in \mathcal{H}_e(3, 3; 5; n - |A| + 1)$.*

With the help of Proposition 2.1 in [21] the authors prove that every graph in $\mathcal{H}_e(3, 3; 4; 18)$ can be obtained by adding 4 independent vertices to a 14-vertex graph H such that $K_1 + H \in \mathcal{H}_e(3, 3; 5; 15)$. All 659 graphs in $\mathcal{H}_e(3, 3; 5; 15)$ were found in [19]. With the help of a computer it was proved that, by extending such 14-vertex graphs H with 4 independent vertices, it is not possible to obtain a graph in $\mathcal{H}_e(3, 3; 4; 18)$, i.e. $\mathcal{H}_e(3, 3; 4; 18) = \emptyset$ and $F_e(3, 3; 4) \geq 19$.

This method is not suitable for proving the Main Theorem, because not all graphs in $\mathcal{H}_e(3, 3; 5; 16)$ are known and their number is too large. Because of this, we first prove that if $G \in \mathcal{H}_e(3, 3; 4; 19)$, then G can be obtained by adding 4 independent vertices to some of 1 139 033 appropriately selected 15-vertex graphs, or by adding 5 independent vertices to some of 113 14-vertex graphs known from [19]. With the help of a computer we check that these extensions do not lead to the construction of a graph in $\mathcal{H}_e(3, 3; 4; 19)$ and we derive that $\mathcal{H}_e(3, 3; 4; 19) = \emptyset$ and $F_e(3, 3; 4) \geq 20$. The computer algorithm which we use (Algorithm 2.4), just as the algorithm used in [21] to prove that $F_e(3, 3; 4) \geq 19$, is based on some ideas of Algorithm A_1 from [19].

For convenience, we will use the following notations:

$$\begin{aligned}\mathcal{L}(n; p) &= \left\{ G : |V(G)| = n, \omega(G) < 4 \text{ and } K_p + G \xrightarrow{e} (3, 3) \right\} \\ \mathcal{L}(n; p; s) &= \{ G \in \mathcal{L}(n; p) : \alpha(G) = s \}\end{aligned}$$

From $R(3, 4) = 9$ it follows that

$$(2.1) \quad \mathcal{L}(n; p; s) = \emptyset, \text{ if } n \geq 9 \text{ and } s \leq 2,$$

and from $R(4, 4) = 18$ it follows that

$$(2.2) \quad \mathcal{L}(n; p; s) = \emptyset, \text{ if } n \geq 18 \text{ and } s \leq 3.$$

In [19] it is proved that $\mathcal{L}(n; 1) \neq \emptyset$ if and only if $n \geq 14$, and all 153 graphs in $\mathcal{L}(14; 1)$ are found. Later, we will use the following fact:

$$(2.3) \quad \mathcal{L}(14; 1; s) \neq \emptyset \Leftrightarrow s \in \{4, 5, 6, 7\}, [19].$$

From (1.1) it follows

$$(2.4) \quad G \in \mathcal{L}(n; p) \Rightarrow \chi(G) \geq 6 - p.$$

Obviously, $\mathcal{H}_e(3, 3; 4; n) = \mathcal{L}(n; 0)$. Let $G \in \mathcal{H}_e(3, 3; 4)$. From the equality $F_e(3, 3; 5) = 15$ [19] and Proposition 2.1 it follows that either $|V(G)| \geq 20$ or $\alpha(G) \leq 5$, and from $R(4, 4) = 18$ it follows that $\alpha(G) \geq 4$. Thus, we obtain

$$(2.5) \quad \mathcal{H}_e(3, 3; 4; 19) = \mathcal{L}(19; 0) = \mathcal{L}(19; 0; 4) \cup \mathcal{L}(19; 0; 5).$$

We will need the following proposition, which follows easily from Proposition 2.1:

Proposition 2.2. *Let $G \in \mathcal{L}(n; p)$, $A \subseteq V(G)$ be an independent set of vertices of G and $H = G - A$. Then $H \in \mathcal{L}(n - |A|; p + 1)$.*

We denote by $\mathcal{L}_{max}(n; p; s)$ the set of all maximal K_4 -free graphs in $\mathcal{L}(n; p; s)$, i.e. the graphs $G \in \mathcal{L}(n; p; s)$ for which $\omega(G + e) = 4$ for every $e \in E(\overline{G})$. Since every graph in $\mathcal{L}(19; 0)$ is contained in a maximal K_4 -free graph, according to (2.5) to prove the Main Theorem it is enough to prove that $\mathcal{L}_{max}(19; 0; 4) = \emptyset$ and $\mathcal{L}_{max}(19; 0; 5) = \emptyset$. In the proofs of these inequalities we will use Algorithm 2.4, formulated below.

The graph G is called a $(+K_3)$ -graph if $G + e$ contains a new 3-clique for every $e \in E(\overline{G})$. We denote by $\mathcal{L}_{+K_3}(n; p; s)$ the set of all $(+K_3)$ -graphs in $\mathcal{L}(n; p; s)$.

Let $G \in \mathcal{L}_{max}(n; p; s)$. Let $A \subseteq V(G)$ be an independent set of vertices of G , $|A| = s$ and $H = G - A$. According to Proposition 2.2, $H \in \mathcal{L}(n - s; p + 1)$. From $\omega(G + e) = 4, \forall e \in E(\overline{G})$ it follows that H is $(+K_3)$ -graph, and from $\alpha(G) = s$ it follows that $\alpha(H) \leq s$. Therefore, $H \in \mathcal{L}_{+K_3}(n - s; p + 1; s')$ for some $s' \leq s$. Thus, we proved the following

Proposition 2.3. *Let $G \in \mathcal{L}_{max}(n; p; s)$. Let $A \subseteq V(G)$ be an independent set of vertices of G , $|A| = s$ and $H = G - A$. Then,*

$$H \in \bigcup_{s' \leq s} \mathcal{L}_{+K_3}(n - s; p + 1; s').$$

The graph G is called a Sperner graph if $N_G(u) \subseteq N_G(v)$ for some pair of vertices $u, v \in V(G)$. If $G \in \mathcal{L}(n; p; s)$ and $N_G(u) \subseteq N_G(v)$, then $G - u \in \mathcal{L}(n - 1; p; s')$, $s - 1 \leq s' \leq s$. Therefore, every Sperner graph $G \in \mathcal{L}(n; p; s)$ is obtained by adding one vertex to some graph $H \in \mathcal{L}(n - 1; p; s')$, $s - 1 \leq s' \leq s$. In the special case, when G is a Sperner graph and $G \in \mathcal{L}_{max}(n; p; s)$, from $N_G(u) \subseteq N_G(v)$ it follows that $N_G(u) = N_G(v)$, and therefore $G - u \in \mathcal{L}_{max}(n - 1; p; s')$, $s - 1 \leq s' \leq s$, i.e. G is obtained by duplicating a vertex in some graph $H \in \mathcal{L}_{max}(n - 1; p; s')$. All non-Sperner graphs in $\mathcal{L}_{max}(n; p; s)$ are obtained with the help of the following algorithm, which is based on Proposition 2.3:

Algorithm 2.4. Finding all non-Sperner graphs in $\mathcal{L}_{max}(n; p; s)$ for fixed n, p and s .

1. Find the set of graphs $\mathcal{A} = \bigcup_{s' \leq s} \mathcal{L}_{+K_3}(n - s; p + 1; s')$. The obtained graphs in $\mathcal{L}_{max}(n; p; s)$ will be output in \mathcal{B} , let $\mathcal{B} = \emptyset$.
2. For each graph $H \in \mathcal{A}$:
 - 2.1. Find the family $\mathcal{M}(H) = \{M_1, \dots, M_t\}$ of all maximal K_3 -free subsets of $V(H)$.
 - 2.2. Find all s -element subsets $N = \{M_{i_1}, M_{i_2}, \dots, M_{i_s}\}$ of $\mathcal{M}(H)$, which fulfill the conditions:
 - (a) $M_{i_j} \neq N_H(v)$ for every $v \in V(H)$ and for every $M_{i_j} \in N$.
 - (b) $K_2 \subseteq H[M_{i_j} \cap M_{i_k}]$ for every $M_{i_j}, M_{i_k} \in N$.
 - (c) If $N' \subseteq N$, then $\alpha(H - \bigcup_{M_{i_j} \in N'} M_{i_j}) \leq s - |N'|$.
 - 2.3. For each s -element subset $N = \{M_{i_1}, M_{i_2}, \dots, M_{i_s}\}$ of $\mathcal{M}(H)$ found in step 2.2 construct the graph $G = G(N)$ by adding new independent vertices v_1, v_2, \dots, v_s to $V(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, \dots, s$. If G is not a Sperner graph and $\omega(G + e) = 4, \forall e \in E(\overline{G})$, then add G to \mathcal{B} .
3. Remove the isomorph copies of graphs from \mathcal{B} .
4. Remove from \mathcal{B} all graphs with chromatic number less than $6 - p$.
5. Remove from \mathcal{B} all graphs G for which $K_p + G \not\rightarrow (3, 3)$.

To avoid repetition in the proofs of Theorem 2.6 and Theorem 4.1 we formulate the following

Lemma 2.5. After the execution of step 2.3 of Algorithm 2.4, the obtained set \mathcal{B} coincides with the set of all maximal K_4 -free non-Sperner graphs with independence number s which have an independent set of vertices $A \subseteq V(G), |A| = s$ such that $G - A \in \mathcal{A}$.

Proof. Suppose that in step 2.3 of Algorithm 2.4 the graph G is added to \mathcal{B} . Let $G = G(N)$ where N and the following notations are the same as in step 2.3. By $H \in \mathcal{A}$, we have $\omega(H) < 4$. Since $N_G(v_j), j = 1, \dots, s$, are K_3 -free sets, it follows that $\omega(G) < 4$. By $H \in \mathcal{A}$, we have $\alpha(H) \leq s$. From this fact and the condition (c) in step 2.2 it follows that $\alpha(G) \leq s$. Since $\{v_1, \dots, v_s\}$ is an independent set of vertices of G , we have $\alpha(G) = s$ and $G - \{v_1, \dots, v_s\} = H \in \mathcal{A}$. The two checks at the end of step 2.3 guarantee that G is a maximal K_4 -free non-Sperner graph.

Let G be a maximal K_4 -free non-Sperner graph with independence number s and $A = \{v_1, \dots, v_s\}$ is an independent set of vertices of G such that $H = G - A \in \mathcal{A}$. We will prove that, after the execution of step 2.3 of Algorithm 2.4, $G \in \mathcal{B}$. Since G is a maximal K_4 -free graph, $N_G(v_i), i = 1, \dots, s$, are maximal K_3 -free subsets of $V(H)$, and therefore $N_G(v_i) \in \mathcal{M}(H), i = 1, \dots, s$ (see step 2.1). Let $N = \{N_G(v_1), \dots, N_G(v_s)\}$. Since G is not a Sperner graph, N is an s -element subset of $\mathcal{M}(H)$ and N fulfills the condition (a) in step 2.2. From the maximality of G it follows that N fulfills the condition (b), and from $\alpha(G) = s$ it follows that N also fulfills (c). Thus, we showed that N fulfills all conditions in step 2.2, and since $G = G(N)$ is a maximal K_4 -free non-Sperner graph, in step 2.3 G is added to \mathcal{B} . \square

Theorem 2.6. After the execution of Algorithm 2.4, the obtained set \mathcal{B} coincides with the set of all non-Sperner graphs in $\mathcal{L}_{max}(n; p; s)$.

Proof. Suppose that, after the execution of Algorithm 2.4, $G \in \mathcal{B}$. According to Lemma 2.5, G is a maximal K_4 -free non-Sperner graph and $\alpha(G) = s$. Now, from step 5 it follows that $G \in \mathcal{L}_{max}(n; p; s)$.

Conversely, let G be an arbitrary non-Sperner graph in $\mathcal{L}_{max}(n; p; s)$. Let $A \subseteq V(G)$ be an independent set of vertices of G , $|A| = s$ and $H = G - A$. According to 2.3, $H \in \mathcal{A}$. Now, from Lemma 2.5 we obtain that, after the execution of step 2.3, the graph G is included in the set \mathcal{B} . By (2.4), after the execution of step 4, G remains in \mathcal{B} . It is clear that after step 5, G also remains in \mathcal{B} . \square

Remark 2.7. In [1] it is proved that if $G \in \mathcal{H}_e(3, 3; 4)$ and $G - e \not\rightarrow (3, 3), \forall e \in E(G)$, then $\delta(G) \geq 8$. Since $\mathcal{L}(18; 0) = \emptyset$, it follows easily that for each graph $G \in \mathcal{L}(19, 0)$ we have $\delta(G) \geq 8$. Using this result we can improve Algorithm 2.4 in the case $n = 19, p = 0$ in the following way:

1. In step 1 we remove from the set \mathcal{A} the graphs with minimum degree less than $8 - s$.

2. In step 2.2 we add the following conditions for the subset N :

(d) $|M_{i_j}| \geq 8$ for every $M_{i_j} \in N$.

(e) If $N' \subseteq N$, then $d_H(v) \geq 8 - s + |N'|$ for every $v \notin \bigcup_{M_{i_j} \in N'} M_{i_j}$.

This way it is guaranteed that in step 2.3 only graphs G for which $\delta(G) \geq 8$ are added to the set \mathcal{B} .

Let us note, however, that in the proof of Theorem 4.1, Algorithm 2.4 must definitely be used without these changes.

The *nauty* programs [14] have an important role in this paper. We use them for fast generation of non-isomorphic graphs and for graph isomorph rejection.

3 Proof of the Main Theorem

According to (2.5) it is enough to prove that $\mathcal{L}_{max}(19; 0; 4) = \emptyset$ and $\mathcal{L}_{max}(19; 0; 5) = \emptyset$.

1. *Proof of $\mathcal{L}_{max}(19; 0; 4) = \emptyset$.*

We generate all 11-vertex graphs with a computer and among them we find all 362 439 graphs in $\mathcal{L}_{+K_3}(11; 2; 3)$ and all 7 949 015 graphs in $\mathcal{L}_{+K_3}(11; 2; 4)$. Let us denote $\mathcal{A}_1 = \mathcal{L}_{+K_3}(11; 2; 3) \cup \mathcal{L}_{+K_3}(11; 2; 4)$. By (2.1),

$$\mathcal{A}_1 = \bigcup_{s' \leq 4} \mathcal{L}_{+K_3}(11; 2; s').$$

We execute Algorithm 2.4 with $n = 15$, $p = 1$, $s = 4$, having $\mathcal{A} = \mathcal{A}_1$ on the first step. According to 2.6, we find all 5750 non-Sperner graphs in $\mathcal{L}_{max}(15; 1; 4)$. Among the graphs in $\mathcal{L}(14; 1)$, which are known from [19], there are 8 maximal K_4 -free graphs and they all have independence number 4. By adding a new vertex to each of these 8 graphs which duplicates some of their vertices, we obtain all 20 non-isomorphic Sperner graphs in $\mathcal{L}_{max}(15; 1; 4)$ (see the text before Algorithm 2.4). Thus, all 5770 graphs in $\mathcal{L}_{max}(15; 1; 4)$ are obtained. All graphs G for which $\omega(G) < 4$ and $\alpha(G) < 4$ are known and can be found in [13]. There are 640 such 15-vertex graphs, among which we find the only 2 graphs in $\mathcal{L}_{max}(15; 1; 3)$. From (2.1) it follows that every graph from the set $\mathcal{A}_2 = \mathcal{L}_{+K_3}(15; 1; 3) \cup \mathcal{L}_{+K_3}(15; 1; 4)$ is a subgraph of some graph in $\mathcal{L}_{max}(15; 1; 3)$ or in $\mathcal{L}_{max}(15; 1; 4)$. By removing edges from the graphs in $\mathcal{L}_{max}(15; 1; 4) \cup \mathcal{L}_{max}(15; 1; 3)$ we obtain all graphs in \mathcal{A}_2 (1 139 028 graphs in $\mathcal{L}_{+K_3}(15; 1; 4)$ and 5 graphs in $\mathcal{L}_{+K_3}(15; 1; 3)$). By (2.1),

$$\mathcal{A}_2 = \bigcup_{s' \leq 4} \mathcal{L}_{+K_3}(15; 1; s').$$

We execute Algorithm 2.4 with $n = 19$, $p = 0$, $s = 4$, having $\mathcal{A} = \mathcal{A}_2$ on the first step. In step 2.3, 2 551 314 graphs are added to the set \mathcal{B} , 2 480 352 of which remain in \mathcal{B} after the isomorph rejection in step 3. After step 4, 2 597 graphs with chromatic number 6 remain in \mathcal{B} . In the end, after executing step 5 we obtain $\mathcal{B} = \emptyset$. From $F_e(3, 3; 4) \geq 19$ it follows that in $\mathcal{L}(19, 0)$ there are no Sperner graphs and by Theorem 2.6 we obtain $\mathcal{L}_{max}(19; 0; 4) = \emptyset$.

2. *Proof of $\mathcal{L}_{max}(19; 0; 5) = \emptyset$.*

From the graphs in $\mathcal{L}(14; 1)$, which are known from [19], with the help of a computer we find all 85 graphs in $\mathcal{L}_{+K_3}(14; 1; 4)$ and all 28 graphs in $\mathcal{L}_{+K_3}(14; 1; 5)$. Let us denote $\mathcal{A}_3 = \mathcal{L}_{+K_3}(14; 1; 4) \cup \mathcal{L}_{+K_3}(14; 1; 5)$. By (2.3),

$$\mathcal{A}_3 = \bigcup_{s' \leq 5} \mathcal{L}_{+K_3}(14; 1; s').$$

We execute Algorithm 2.4 with $n = 19$, $p = 0$, $s = 5$, having $\mathcal{A} = \mathcal{A}_3$ on the first step. In step 2.3, 502 901 graphs are added to the set \mathcal{B} , 251 244 of which remain in \mathcal{B} after the isomorph rejection in step 3. After step 4, 31 graphs with chromatic number 6 remain in \mathcal{B} . In the end, after executing step 5 we obtain $\mathcal{B} = \emptyset$. Since there are no Sperner graphs in $\mathcal{L}(19; 0)$, by Theorem 2.6 we obtain $\mathcal{L}_{max}(19; 0; 5) = \emptyset$. \square

All computations were done on a personal computer. Using one processing core, the time needed to execute Algorithm 2.4 in the case $n = 19$, $p = 0$, $s = 4$ is about one hour, and in the case $n = 19$, $p = 0$, $s = 5$ it is about half a minute. Note that in the first case among the 2597 6-chromatic graphs obtained after step 4 of Algorithm 2.4, 794 have a minimum degree of 8 or more. In the second case, the total number of 6-chromatic graphs is 31, 11 of which have a minimum degree of 8 or more. Using the improvements of Algorithm 2.4 described in Remark 2.7, the time needed for computations is reduced almost 2 times in the first case and more than 10 times in the second case.

Remark 3.1. *Let us note that the result $\mathcal{L}_{max}(19; 0; 5) = \emptyset$ can also be obtained with the algorithm from [21], but more computations have to be performed by the computer.*

4 New bounds on the vertex Folkman number

$$F_v(2, 3, 3; 4)$$

Let G be a graph and a_1, \dots, a_s be positive integers. The expression $G \xrightarrow{v} (a_1, \dots, a_s)$ means that in every coloring of $V(G)$ in s colors there exist $i \in \{1, \dots, s\}$ such that there is a monochromatic a_i -clique of color i .

Define:

$$\mathcal{H}_v(a_1, \dots, a_s; q) = \left\{ G : G \xrightarrow{v} (a_1, \dots, a_s) \text{ and } \omega(G) < q \right\}.$$

$$F_v(a_1, \dots, a_s; q) = \min \{ |V(G)| : G \in \mathcal{H}_v(a_1, \dots, a_s; q) \}.$$

The numbers $F_v(a_1, \dots, a_s; q)$ are called vertex Folkman numbers. The edge Folkman numbers $F_e(a_1, \dots, a_s; q)$ are defined similarly.

Folkman [5] proved that $F_v(a_1, \dots, a_s; q)$ exists if and only if $q > \max \{a_1, \dots, a_s\}$. In this section, we consider the numbers $F_v(3, 3; q)$ and $F_v(2, 3, 3; q)$, which are related to the edge Folkman number $F_e(3, 3; 4)$. According to Folkman's result, these two numbers exist if and only if $q \geq 4$.

All the numbers $F_v(3, 3; q)$ are known:

$$F_v(3, 3; q) = \begin{cases} 5, & \text{if } q \geq 6 \text{ (obvious)} \\ 8, & \text{if } q = 5 \\ 14, & \text{if } q = 4, [15] \text{ and } [19] \end{cases}$$

About the numbers $F_v(2, 3, 3; q)$ it is known that:

$$F_v(2, 3, 3; q) = \begin{cases} 6, & \text{if } q \geq 7 \text{ (obvious)} \\ 9, & \text{if } q = 6 \\ 12, & \text{if } q = 5, [18] \end{cases}$$

The equalities $F_v(3, 3; 5) = 8$ and $F_v(2, 3, 3; 6) = 9$ are a special case of Theorem 3 [12].

The number $F_v(2, 3, 3; 4)$ is not known and its computation seems to be quite difficult. The following bounds are known:

$$(4.1) \quad 19 \leq F_v(2, 3, 3; 4) \leq 30, [25] \text{ and } [24].$$

Using the results from this paper (the graphs obtained in the proof of the Main Theorem), we improve the bounds in (4.1) by proving the following

Theorem 4.1. $20 \leq F_v(2, 3, 3; 4) \leq 24$.

The interest in the number $F_v(2, 3, 3; 4)$ is motivated by our conjecture that the inequality $F_v(2, 3, 3; 4) \leq F_e(3, 3; 4)$ is true.

If this inequality holds, then there would be another possible way to prove a lower bound on the number $F_e(3, 3; 4)$ (the number $F_v(2, 3, 3; 4)$ is easier to bound than $F_e(3, 3; 4)$).

Later we will use the following obvious fact:

$$(4.2) \quad G \xrightarrow{v} (2, 3, 3) \Rightarrow \chi(G) \geq 6.$$

Define:

$$\mathcal{H}_v(a_1, \dots, a_s; q; n) = \{G : G \in \mathcal{H}_v(a_1, \dots, a_s; q) \text{ and } |V(G)| = n\}.$$

Let $G \xrightarrow{v} (2, 3, 3)$ and $A \subseteq V(G)$ be an independent set of vertices. Then, obviously, $G - A \xrightarrow{v} (3, 3)$. Therefore, the following proposition is true:

Proposition 4.2. *Let $G \in \mathcal{H}_v(2, 3, 3; 4; n)$ and $A \subseteq V(G)$ be an independent set of vertices of G . Then, $G - A \in \mathcal{H}_v(3, 3; 4; n - |A|)$.*

It is easy to see that if $G \xrightarrow{v} (3, 3)$, then $K_1 + G \xrightarrow{e} (3, 3)$. Thus, we derive

Proposition 4.3. $\mathcal{H}_v(3, 3; 4; n) \subseteq \mathcal{L}(n; 1)$.

Remark 4.4. *Let us note, that $\mathcal{H}_v(3, 3; 4; 14) = \mathcal{L}(14; 1)$ [19], but $\mathcal{H}_v(3, 3; 4; 15) \neq \mathcal{L}(15; 1)$. We found all 2 081 234 graphs $\mathcal{L}(15; 1)$. In the proof of the Main Theorem we already found the only 2 graphs in $\mathcal{L}_{\max}(15; 1; 3)$ and all 5770 graphs in $\mathcal{L}_{\max}(15; 1; 4)$. With the help of Algorithm 2.4, similarly to the case $s = 4$, we determine $\mathcal{L}_{\max}(15; 1; s)$, $s \geq 5$. We obtain all 826 graphs $\mathcal{L}_{\max}(15; 1; 5)$, all 12 graphs in $\mathcal{L}_{\max}(15; 1; 6)$ and $\mathcal{L}_{\max}(15; 1; s) = \emptyset$, $s \geq 7$. Thus, we obtain all 6 610 maximal K_4 -free graphs in $\mathcal{L}(15; 1)$. By removing edges from them, we find all 2 081 234 graphs in $\mathcal{L}(15; 1)$. Some properties of these graphs are listed in Table 1. Among the graphs in $\mathcal{L}(15; 1)$ there are exactly 20 graphs, which are not in $\mathcal{H}_v(3, 3; 4; 15)$. Properties of these 20 graphs are given in Table 1, and one of these graphs (which has 51 edges) is shown in Figure 1.*

$ E(G) $	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$ Aut(G) $	#
42	1	0	153	7	65	3	5	1	2 052 543
43	4	1	1 629	8	675 118	4	1 300 452	2	27 729
44	44	2	10 039	9	1 159 910	5	747 383	3	9
45	334	3	34 921	10	165 612	6	32 618	4	850
46	2 109	4	649 579	11	80 529	7	766	6	22
47	9 863	5	1 038 937			8	10	8	55
48	35 812	6	339 395					10	2
49	101 468	7	6 581					12	11
50	223 881							14	4
51	378 614							16	4
52	478 582							20	1
53	436 693							24	4
54	273 824								
55	110 592								
56	26 099								
57	3 150								
58	160								
59	4								

Table 1: Statistics of the graphs in $\mathcal{L}(15;1)$

$ E(G) $	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$ Aut(G) $	#
47	2	4	7	8	20	4	5	1	5
48	5	5	10			5	15	2	12
49	7	6	3					4	3
50	3								
51	1								
52	2								

Table 2: Statistics of the graphs in $\mathcal{L}(15;1) \setminus \mathcal{H}_v(3,3;5;15)$

```

0 0 0 1 1 1 1 0 0 0 0 1 0 1 0
0 0 0 1 1 1 1 0 0 0 0 0 1 0 1
0 0 0 0 0 0 0 1 1 0 0 1 1 1 1
1 1 0 0 0 0 0 0 1 1 0 1 1 0 0
1 1 0 0 0 0 0 1 0 0 1 1 1 0 0
1 1 0 0 0 0 0 1 0 1 0 0 0 1 1
1 1 0 0 0 0 0 1 0 1 0 0 0 1 1
1 1 0 0 0 0 0 1 0 1 0 0 0 1 1
0 0 1 0 1 0 1 0 0 1 1 1 0 0 1
0 0 1 1 0 1 0 0 0 1 1 0 1 1 0
0 0 0 1 0 0 1 1 1 0 1 1 0 1 0
0 0 0 0 1 1 0 1 1 1 0 0 1 0 1
1 0 1 1 1 0 0 1 0 1 0 0 1 1 0
0 1 1 1 1 0 0 0 1 0 1 1 0 0 1
1 0 1 0 0 1 1 0 1 1 0 1 0 0 1
0 1 1 0 0 1 1 1 0 0 1 0 1 1 0

```

Figure 1: Example of a graph in $\mathcal{L}(15;1) \setminus \mathcal{H}_v(3,3;5;15)$

```

0 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
1 0 1 1 0 0 0 0 0 1 0 0 1 1 0 0 1 1 1 1 0 0 0 0
1 1 0 0 0 0 1 0 0 0 1 0 1 0 1 0 1 0 0 1 0 1 1 0
1 1 0 0 0 0 0 1 0 0 1 0 0 0 1 1 1 0 1 0 1 0 0 1
1 0 0 0 0 0 1 0 1 0 1 0 1 1 0 0 0 1 1 0 0 1 0 1
1 0 0 0 0 0 0 1 1 0 1 0 0 1 0 1 0 1 0 1 1 0 1 0
1 0 1 0 1 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 1 1 0 1
1 0 0 1 0 1 0 0 0 1 0 0 1 0 1 0 0 0 1 0 1 1 1 0
1 0 0 0 1 1 0 0 0 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1
1 1 0 0 0 0 1 1 1 0 0 1 0 0 0 0 0 0 1 1 0 0 1 1
1 0 1 1 1 1 0 0 0 0 0 1 0 0 0 0 1 1 0 0 1 1 0 0
0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 1 1 1 1 1 1 1
0 1 1 0 1 0 0 1 0 0 0 0 0 1 1 0 0 1 0 0 1 0 1 1
0 1 0 0 1 1 1 0 0 0 0 0 1 0 0 1 1 0 1 0 1 0 1 0
0 0 1 1 0 0 0 1 1 0 0 0 1 0 0 1 0 1 0 1 0 1 0 1
0 0 0 1 0 1 1 0 1 0 0 0 0 1 1 0 1 0 1 1 0 1 0 0
0 1 1 1 0 0 0 0 1 0 1 1 0 1 0 1 0 0 0 0 0 0 1 1
0 1 0 0 1 1 0 0 1 0 1 1 1 0 1 0 0 0 1 1 0 0 0 0
0 1 0 1 1 0 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0 1 0 0
0 1 1 0 0 1 1 0 0 1 0 1 0 0 1 1 0 1 0 0 1 0 0 0
0 0 0 1 0 1 1 1 0 0 1 1 1 1 0 0 0 0 0 1 0 0 0 1
0 0 1 0 1 0 1 1 0 0 1 1 0 0 1 1 0 0 1 0 0 0 1 0
0 0 1 0 0 1 0 1 1 1 0 1 1 1 0 0 1 0 0 0 0 1 0 0
0 0 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 0 0 0 1 0 0 0

```

Figure 2: 24-vertex transitive graph in $\mathcal{H}_v(2, 3, 3; 4)$

Proof of theorem 4.1

1. Proof of the inequality $F_v(2, 3, 3; 4) \geq 20$.

It is enough to prove that $\mathcal{H}_v(2, 3, 3; 4; 19) = \emptyset$. Suppose the opposite is true and let $G \in \mathcal{H}_v(2, 3, 3; 4; 19)$ be a maximal K_4 -free graph. Since $\mathcal{H}_v(2, 3, 3; 4; 18) = \emptyset$, G is not a Sperner graph. From Proposition 4.2 and $F_v(3, 3; 4) = 14$ it follows that $\alpha(G) \leq 5$, and from $R(4, 4) = 18$ it follows that $\alpha(G) \geq 4$. Therefore, the following two cases are possible:

Case 1: $\alpha(G) = 4$. Let $A \subseteq V(G)$ be an independent set of vertices of G , $|A| = 4$ and $H = G - A$. From Proposition 4.2 it follows that $H \in \mathcal{H}_v(3, 3; 4; 15)$ and from Proposition 4.3 we obtain $H \in \mathcal{L}(14; 1; s)$, $s \leq 4$. Since G is maximal K_4 -free graph, it follows that

$$(4.3) \quad H \in \mathcal{L}_{+K_3}(15; 1; s), s \leq 4.$$

We execute Algorithm 2.4 ($n = 19, p = 0, s = 4$). According to (4.3), $H \in \mathcal{A}$, and therefore, from Lemma 2.5 we obtain that, after the execution of step 2.3 of the algorithm, $G \in \mathcal{B}$. By (4.2), $G \in \mathcal{B}$ after step 4. On the other hand, in the first part of the proof of the Main Theorem we found all 2 597 graphs which are in \mathcal{B} after step 4. None of these graphs belongs to $\mathcal{H}_v(2, 3, 3; 4)$, which is a contradiction.

Case 2: $\alpha(G) = 5$. In the same way as in the proof of Case 1, we see that after the execution of step 4 of Algorithm 2.4 $G \in \mathcal{B}$. In the second part of the proof of the Main Theorem we found all 31 graphs in \mathcal{B} . None of these graphs belongs to $\mathcal{H}_v(2, 3, 3; 4)$, which is a contradiction.

2. Proof of the inequality $F_v(2, 3, 3; 4) \leq 24$.

All vertex transitive graphs with up to 31 are known and can be found in [23]. With the help of a computer we check which of these graphs belong to $\mathcal{H}_v(2, 3, 3; 4)$. In this way, we find one 24-vertex graph, one 28-vertex graph and 6 30-vertex graphs in $\mathcal{H}_v(2, 3, 3; 4)$. The only 24-vertex transitive graph in $\mathcal{H}_v(2, 3, 3; 4)$ is given on Figure 2. It does not have proper subgraphs in $\mathcal{H}_v(2, 3, 3; 4)$, but by adding edges to this graph we obtain 18 more graphs in $\mathcal{H}_v(2, 3, 3; 4; 24)$, two of which are maximal K_4 -free graphs.

References

- [1] A. Bikov. Small minimal $(3, 3)$ -Ramsey graphs. preprint: arxiv:1604.03716, April 2016.
- [2] A. Dudek and V. Rödl. On the Folkman Number $f(2, 3, 4)$. *Experimental Mathematics*, 17:63–67, 2008.
- [3] P. Erdős. Problems and results on finite and infinite graphs. Recent Advances in Graph Theory *Proc. Second Czechoslovak Sympos.*, Prague, 1974, 183–192, Academia, Prague, 1975.
- [4] P. Erdős and A. Hajnal. Research problem 2-5. *J. Combin. Theory*, 2:104, 1967.
- [5] J. Folkman. Graphs with monochromatic complete subgraph in every edge coloring. *SIAM J. Appl. Math.*, 18:19–24, 1970.
- [6] P. Frankl and V. Rödl. Large triangle-free subgraphs in graphs without K_4 . *Graphs and Combinatorics*, 2:135–144, 1970.
- [7] R. L. Graham. On edgewise 2-colored graphs with monochromatic triangles containing no complete hexagon. *J. Combin. Theory*, 4:300, 1968.
- [8] R. L. Graham. Some Graph Theory Problems I Would Like to See Solved. *SIAM My Favorite Graph Theory Conjectures*, Halifax, 2012.
- [9] A. Lange, S. Radziszowski, and X. Xu. Use of MAX-CUT for Ramsey Arrowing of Triangles. *Journal of Comb. Math. and Comb. Comp.*, 88:61–71, 2014.
- [10] S. Lin. On Ramsey numbers and K_r -coloring of graphs. *J. Combin. Theory Ser. B*, 12:82–92, 1972.
- [11] L. Lu. Explicit Construction of Small Folkman Graphs. *SIAM J. on Discrete Math.*, 21:1053–1060, 2008.
- [12] T. Luczak, A. Ruciński, and S. Urbański. On minimal vertex Folkman graphs. *Discrete Mathematics*, 236:245–262, 2001.
- [13] B.D. McKay. <http://cs.anu.edu.au/~bdm/data/ramsey.html>.
- [14] B. D. McKay and A. Piperino. Practical graph isomorphism, II. *J. Symbolic Computation*, 60:94–112, 2013. Preprint version at arxiv.org.

- [15] N. Nenov. An example of a 15-vertex Ramsey $(3, 3)$ -graph with clique number 4. (in Russian). *C. A. Acad. Bulg. Sci.*, 34:1487–1489, 1981.
- [16] N. Nenov. On the Zykov numbers and some its applications to Ramsey theory. (in Russian). *Serdica Bulg. math. publ.*, 9:161–167, 1983.
- [17] N. Nenov. The chromatic number of any 10-vertex graph without 4-cliques is at most 4. (in Russian). *C. A. Acad. Bulg. Sci.*, 37:301–304, 1984.
- [18] N. Nenov. Computation of the vertex Folkman numbers $F(2, 2, 2, 3; 5)$ and $F(2, 3, 3; 5)$. *Ann. Univ. Sofia Fac. Math. Inform.*, 95:71–82, 2001.
- [19] K. Piwakowski, S. Radziszowski, and S. Urbański. Computation of the Folkman number $F_e(3, 3; 5)$. *J. Graph Theory*, 32:41–49, 1999.
- [20] S. Radziszowski. Small Ramsey numbers. *The Electronic Journal of Combinatorics*, Dynamic Survey revision 14, January 12 2014.
- [21] S. Radziszowski and X. Xu. On the Most Wanted Folkman Graph. *Geombinatorics*, XVI(4):367–381, 2007.
- [22] S. Radziszowski and X. Xu. On Some Open Questions for Ramsey and Folkman Numbers. To appear in *Graph Theory, Favorite Conjectures and Open Problems*, 2014.
- [23] G. Royle. <http://staffhome.ecm.uwa.edu.au/~00013890/trans/>.
- [24] Z. Shao, M. Liang, J. He, and X. Xu. New Lower Bounds for Two Multicolor Vertex Folkman Numbers. *International Conference on Computer and Management (CAMAN)*, Wuhan, China, 1-3, 2011.
- [25] Z. Shao, X. Xu, and H. Luo. Bounds for two multicolor vertex Folkman numbers (in Chinese). *Application Research of Computers*, 26:834–835, 2009.
- [26] A. Soifer. *The Mathematical Coloring Book*. Springer, 2008.
- [27] J. Spencer. *Three hundred million points suffice*. *J. Combin. Theory Ser. A*, 49:210–217, 1998. Also see erratum by M. Hovey in 50:323.
- [28] D. West. *Introduction to Graph Theory*. Prentice Hall, Inc., Upper Saddle River, 2 edition, 2001.